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The Orthogonal Rational Functions of Higgins and Christov and Algebraically Mapped Chebyshev Polynomials

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It is shown that the rational functions of Higgins and Christov, orthogonal on $[-\infty, \infty]$, are Chebyshev polynomials of the first and second kinds with an algebraic change of variable. Because of these relationships, the existing theory and algorithms for mapped Chebyshev polynomials also apply to the rational functions: the Higgins and Christov functions have excellent numerical properties. However —precisely because of these same connections—it is usually simpler to use the change of variable rather than write computer programs that employ the Higgins and Christov functions themselves. Nonetheless, the result is a series of orthogonal rational functions. For some problems whose solutions decay slowly (algebraically rather than exponentially with $|y|$), such as the "Yoshida jet" in oceanography, a Christov expansion is the only spectral series that converges rapidly. © 1990 Academic Press, Inc.

1. INTRODUCTION

The book by Higgins [1] and the paper by Christov [2] discuss two interesting basis sets that, in contrast to the familiar orthogonal *polynomials*, are *rational* functions, orthogonal on $[-\infty, \infty]$. Boyd [3] constructed a similar basis set by applying an algebraic map to the Chebyshev polynomials. This note extends the prior work of Higgins and Christov by giving convergence theorems and a numerical methodology for their functions and by showing the relationship between their basis sets and those of Boyd.

Section 2 establishes the main theorems, which express the connections between the Higgins and Christov functions, the Chebyshev polynomials, and the terms of an ordinary Fourier series. Although the theorems are proved by elementary algebra, previous workers were unaware of them. Once demonstrated, these connections allow one to borrow the existing

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numerical analysis for mapped Chebyshev polynomials and to apply it to the Higgins and Christov functions with only trivial modifications as discussed in Sect. 3 and in Boyd [6]. The final section is a summary and prospectus.

2. Results

We begin with some definitions. Note that what we describe as "Higgins" functions are given in [1] only in complex form, and that our notation differs from Christov's.

Definition 1. The "rational Chebyshev functions of the first kind" are defined by (Boyd [3], but without the notation introduced here

$$TB_n(y) \equiv T_n[y/(1+y^2)^{1/2}], \tag{1}$$

where the $T_n(x)$ are the usual Chebyshev polynomials [4]. Note that the $TB_n(y)$ are rational functions for n even and rational functions divided by $(1+y^2)^{1/2}$ when n is odd.

Definition 2. The "rational Chebyshev functions of the second kind" are

$$UB_n(y) \equiv U_n[y/(1+y^2)^{1/2}], \tag{2}$$

where the $U_n(x)$ are the standard Chebyshev polynomials of the second kind.

Like the $TB_n(y)$, the $UB_n(y)$ are rational functions only when n is even, i.e., when the function is symmetric about $y=0$. but in a minor abuse of terminology, we shall refer to these, for all n, as "rational Chebyshev functions."

Definition 3. The "Christov functions" are defined by

$$CC_{2n}(y) = [\mu_n(y) - \mu_{-n-1}(y)]/2 \qquad n = 0, 1, 2, \ldots \tag{3}$$

$$SC_{2n+1}(y) = -[\mu_n(y) + \mu_{-n-1}(y)]/(2i) \qquad n = 0, 1, 2, \ldots, \tag{4}$$

where the "complex Christov functions" $\mu_n(y)$ are

$$\mu_n(y) \equiv (iy-1)^n/(iy+1)^{n+1} \qquad n = 0, \pm 1, \pm 2, \ldots . \tag{5}$$

The $CC_{2n}(y)$, like the $TB_{2n}(y)$ and the $UB_{2n}(y)$, are symmetric about $y=0$; the initial "C" in the symbol was introduced by Christov as a reminder that these functions are "cosine-like" in the sense of having this symmetry. Similarly, the SC_{2n+1} are "sine-like" in that they are antisymmetric about the origin, just like the odd degree rational Chebyshev functions. In contrast to the latter, however, both the CC_{2n} and SC_{2n+1} lack a square root factor and are rational functions for all n.

DEFINITION 4. The "Higgins functions" are

$$CH_{2n}(y) \equiv [\lambda_n(y) + \lambda_{-n}(y)]/2 \qquad n = 0, 1, 2, \ldots \tag{6}$$

$$SH_{2n+1}(y) \equiv [\lambda_{n+1}(y) - \lambda_{-n-1}(y)]/(2i) \qquad n = 0, 1, 2, \ldots, \tag{7}$$

where the "complex Higgins functions" are

$$\lambda_n(y) \equiv (iy - 1)^n/(iy + 1)^n \qquad n = 0, \pm 1, \pm 2. \tag{8}$$

The sine- and cosine-like Higgins functions have not been previously defined.

To prove Theorem 1 below, we must first establish a lemma which shows that the Chebyshev functions can be written in a form that mimics the definitions of the Higgins and Christov functions.

LEMMA 1. *The Chebyshev rational functions can be written as the sum or difference of a pair of complex functions, viz.,*

$$TB_n(y) = [\sigma_n(y) + \sigma_{-n}(y)]/2 \qquad n = 0, 1, 2, \ldots, \tag{9}$$

where

$$\sigma_n(y) \equiv (iy - 1)^{n/2}/(iy + 1)^{n/2} \qquad n = 0, \pm 1, \pm 2, \ldots \tag{10}$$

and

$$UB_n(y) = [\tau_{-n-2}(y) - \tau_n(y)]/2 \qquad n = 0, 1, 2, \ldots, \tag{11}$$

where

$$\tau_n(y) \equiv (iy - 1)^{n/2 + \frac{1}{2}}/(iy + 1)^{n/2} \qquad n = 0, \pm 1, \pm 2, \ldots \tag{12}$$

Proof. Davis [5, pg. 83] proves that the Chebyshev polynomials have the representation

$$T_n(x) = [z(x)^n + z(x)^{-n}]/2 \tag{13}$$

where [5, pp. 19–20] the mapping $x = (z + 1/z)/2$ has the explicit inverse

$$z(x) = x + (x^2 - 1)^{1/2}. \tag{14}$$

The map that transforms $T_n(x) \to \mathrm{TB}_n(y)$ is

$$x = y/(1+y^2)^{1/2}. \tag{15}$$

Substituting this map in (14) then shows

$$z(y) = (y+i)/(1+y^2)^{1/2} = (y+i)^{1/2}/(y-i)^{1/2}. \tag{16}$$

It is then trivial to see that (13) is identical with (9) if we use the definition (10).

The proof of (11) is similar. We begin by using the well-known identity [4]

$$U_n(x) = [1/(n+1)]\, dT_{n+1}(x)/dx \qquad n = 0, 1, 2, \ldots . \tag{17}$$

By applying this to (13) and then using (14)–(17), we obtain a representation of the $\mathrm{UB}_n(y)$ identical to (11). Q.E.D.

THEOREM 1. *The Christov and Higgins functions are related to the Chebyshev rational functions as*

[Christov]

$$\mathrm{CC}_{2n}(y) = [1/(1+y^2)]\, U_{2n}[y/(1+y^2)^{1/2}] \tag{18}$$

$$\mathrm{SC}_{2n+1} = [1/(1+y^2)]^{1/2}\, T_{2n+1}[y/(1+y^2)^{1/2}] \tag{19}$$

[Higgins]

$$\mathrm{CH}_{2n}(y) = T_{2n}[y/(1+y^2)^{1/2}] \tag{20}$$

$$\mathrm{SH}_{2n+1}(y) = [1/(1+y^2)]^{1/2}\, U_{2n+1}[y/(1+y^2)^{1/2}]. \tag{21}$$

Proof. Comparison of the identities in Lemma 1 with Definitions 3 and 4. For example, inspection of (8) and (10) shows that $\sigma_{2n}(y) \equiv \lambda_n(y)$. Comparison of (6) with (9) then establishes (20). The other three parts of the theorem are proved in exactly the same way. Q.E.D.

It is well known that the Chebyshev polynomials are merely trigonometric functions with a change of variable, *viz.*, $T_n(\cos t) \equiv \cos(nt)$ and $U_n(\cos t) \equiv \sin([n+1]t)/\sin(t)$. The rational orthogonal functions have similar equivalents as expressed by the following.

THEOREM 2. *With the mapping*

$$y = \cot(t) \leftrightarrow t = \operatorname{arccos}(y) \tag{22}$$

$$\mathrm{TB}_n(y) = \cos(nt) = \cos(n \operatorname{arccot}[y]) \tag{23}$$

$$\mathrm{UB}_n(y) = \sin([n+1]t)/\sin(t) \tag{24}$$

$$\mathrm{CC}_{2n}(y) = (\cos[2nt] - \cos[(2n+2)t])/2 \tag{25}$$

$$\mathrm{SC}_{2n+1}(y) = (\sin[(2n+2)t] - \sin[2nt])/2 \tag{26}$$

$$\mathrm{CH}_{2n}(y) = \cos(2nt) \tag{27}$$

$$\mathrm{SH}_{2n+1}(y) = \sin[(2n+2)t]. \tag{28}$$

Proof. Use of the trigonometric definitions of $T_n(x)$ and $U_n(x)$ plus Theorem 1.

3. IMPLICATIONS

The close relationship between the Christov and Higgins functions and the Chebyshev rational functions implies that many good features of the latter [6] carry over automatically to the former. First, the Christov and Higgins expansions will have coefficients that decrease *exponentially* fast with n for any function $f(y)$ that is analytic along the whole real axis except at infinity where it has bounded derivatives.

Second, the functions and their derivatives can be computed through simple recurrences. Christov [2] derives many such formulas for his basis set, but his method—based on obtaining relations for the complex functions $\mu_n(y)$ first—is applicable to all the others including the rational Chebyshev functions. In particular, we note that all four species of basis functions satisfy the same recurrence,

$$\phi_{m+2}(y) = 2\{[y^2 - 1]/[y^2 + 1]\}\,\phi_m(y) - \phi_{m-2}(y), \tag{29}$$

where $\{\phi_m(y)\}$ is any of TB_{2n}, TB_{2n+1}, UB_{2n}, UB_{2n+1}, CC_{2n}, SC_{2n+1}, CH_{2n}, SH_{2n+1} with the appropriate starting values. The reason for (29) is that the Chebyshev polynomials of both kinds satisfy the same recurrence. The extra factors of $(1 + y^2)$ in the Christov and Higgins functions merely alter the starting values from those for the $\mathrm{TB}_n(y)$ and $\mathrm{UB}_n(y)$.

Third, as noted by Christov [2], the derivative of a rational orthogonal function of any of the classes discussed here is the sum of at most three basis functions. This implies that when we use Galerkin's method to convert a differential equation into a matrix problem, the matrix is highly banded with only a handful of nonzero matrix elements in each row.

Thus, the Higgins and Christov functions are an entirely practical basis for the numerical solutions of differential equations on $y \in [-\infty, \infty]$.

Unfortunately, numerical experience—the examples in [2] and in [6], and earlier works of the author—has shown that the most practical way of using these orthogonal rational functions is to simply change the variable from $y \in [-\infty, \infty]$ to $t \in [0, \pi]$ as given by (22). For example, the parabolic cylinder equation

$$u_{yy} + [E - y^2]u = 0 \tag{30}$$

where E is the eigenvalue becomes

$$\sin^6(t)\, u_{tt} + 2\cos(t)\sin^5(t)\, u_t + [E\sin^2(t) - \cos^2(t)]u = 0. \tag{31}$$

Instead of deriving many new identities in y as Christov has done, we can use the familiar *trigonometric* identities to show that after decoupling the matrix problem for the symmetric eigenfunctions from that for the antisymmetric modes, each matrix is heptadiagonal—that is, there are only seven nonzero elements in each row. Boyd [6] gives a table of the cosine-collocation solution of (31).

For functions like $\mathrm{sech}(y)$ or $\exp(-0.5\, y^2)$ which decay exponentially with y as $|y| \to \infty$, the differences between the four basis sets are negligible. Christov's functions, which themselves decay with $|y|$ and thus *individually* satisfy the boundary condition, would *appear* to be preferable. However, (30) is *singular* at ∞. The solution which decays for large $|y|$ is the only solution which is even bounded at ∞. The numerical results of Boyd [6] show that the collocation method automatically converges to this bounded solution even when an *unconstrained* set of $TB_n(y)$ is used as the expansion. In the language of Boyd [7], the condition of boundedness at ∞ is a "natural" rather than "essential" boundary condition, and it is not necessary to impose it on the individual basis functions.

For solutions which decay *algebraically* with y, however, it is a different story. The Chebyshev rational functions give exponential convergence with n provided the decay is as an *even* power of y, but inspecting the set $\{\cos(nt)\}$ and the map $y = \cot(t)$ in the limit $t \to 0$ (that is, as $y \to \infty$) shows that it is not possible to use the cosines—or even linear combinations of the cosines—to match the behavior of a function which decays as an *odd* power of y; for example,

$$f(y) \equiv y/(1 + y^2) = \sin(2t). \tag{32}$$

If we expand this as a cosine series in t—a series of TB_n in y—we find that the coefficients are $O(1/n^2)$. The Christov expansion, in contrast, converges in just one term because (32) is $SC_1(y)$! Christov [2] and Boyd [6] give physical examples whose solutions fall in this same class: antisymmetric about the origin with decay as $1/y$. One can show [6] that the coefficients of the Christov series decrease exponentially with n while those for the rational Chebyshev expansion are still $O(1/n^2)$. Thus, we conclude that the Chebyshev basis functions are poor for some functions that decay algebraically with y.

However, the machinery of Christov's recursions is still unnecessary: to expand functions like (32), we can simply change variables and use a sine series in t. It is easy to invent examples such as $g(y) = (1 + y^2)^{1/2}$ which are *symmetric* about the origin and decay as an odd power of $|y|$. All four of the basis sets discussed here fail for this class of functions because all the sets are equivalent (for a symmetric $g(y)$) to a *cosine* series in t, but the remedy is to simply change variables and use a Fourier *sine* series.

4. Summary

In this note, we have shown that the orthogonal rational functions of Higgins and Christov have excellent numerical properties. In particular, if a function $f(y)$ decays exponentially as $|y| \to \infty$ along the real axis and the function has no singularities for real y except at infinity, then the terms in a series of Higgins or Christov functions will decrease exponentially fast with n. The rational function series may also give exponential convergence for $f(y)$ which decay algebraically with $|y|$ or tend to a constant as $|y| \to \infty$, provided that $f(y)$ has an asymptotic series in $1/y$ of the appropriate form. Because of their close connection with both sines and cosines and with the Chebyshev polynomials, it is easy to apply both Galerkin and pseudospectral methods with the Higgins and Christov functions, and the former will give banded matrices if the differential equation has polynomial or rational coefficients.

Boyd [8] shows that one may define similar orthogonal rational functions for the *semi-infinite* interval, $y \in [0, \infty]$. These functions, denoted $TL_n(y)$, have no direct connection with the basis sets on $[-\infty, \infty]$ discussed here, but have many similar properties including a simple connection (via a mapping transformation) with a Fourier cosine series.

The only *caveat* about orthogonal rational functions is that precisely because of these same connections with trigonometric functions, it is usually easier to apply the change of variable (22) and then use the equivalent Fourier series method than it is to work with the Higgins or Christov functions explicitly. For this reason, the theorems derived here have been omitted from the applications-oriented article [6] in favor of this self-contained treatment.

Nonetheless, this change-of-variable strategy for writing the *computer program* does not alter the fact that $f(y)$ is approximated by a sum of *rational* functions. Boyd [6] shows that for the problem of the so-called "Yoshida jet" in oceanography, the asymptotic behavior of $v(y)$ (antisymmetric about $y = 0$ with an asymptotic series in odd powers of $1/y$) is such that Hermite series, sinc expansions, and sums of algebraically mapped Chebyshev polynomials ($TB_n(y)$) all converge very, very slowly (as some small inverse power of the number of terms in the series). In contrast, the Christov expansion is exponentially convergent. Even if the pseudospectral matrices were created by evaluating trigonometric functions (as was in fact the case), the approximation [6]

$$v = -y(440.8017 + 15.098\, y^2 + 1.1412\, y^4)/(9 + y^2)^3, \tag{33}$$

which has a maximum error of only 0.006 for $y \in [-\infty, \infty]$, is merely the sum of the first three terms in the Christov series in $(y/3)$. Nothing else works well.

The conclusion is that even though a programmer need not become an expert in their properties, the Higgins and Christov functions have a small but secure place in the numerical toolbox.

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Note added in proof. Christov and Bekyarov [9] have successfully applied Christov functions to compute solitary waves of the Korteweg–de Vries and Kuramoto–Sivashinsky equations. Orthogonal rational functions are also discussed in the new book by the author [10].

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